ON INTEGER SOLUTIONS TO $a^2 + b^3 + c^4 = d^5$

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Abstract. We consider Diophantine equation $a^2 + b^3 + c^4 = d^5$ and related equations.

1. The equation $a^2 + b^3 + c^4 = d^5$ and similar equations

We first consider the equation

$$a^2 + b^3 + c^4 = d^5. \tag{1.1}$$

Some solutions:

$$2^2 + 3^3 + 1^4 = 2^5 \tag{1.2}$$

$$9^2 + 38^3 + 8^4 = 9^5 \tag{1.3}$$

$$16^2 + 8^3 + 4^4 = 4^5 \tag{1.4}$$

$$24^2 + 31^3 + 7^4 = 8^5 \tag{1.5}$$

$$41^2 + 31^3 + 6^4 = 8^5 \tag{1.6}$$

Every particular solution may be made into a family by adding a coefficient u^{k_i} to the bases. For instance, solution (1.2) generates a family of solutions

$$\left(2u^{30}\right)^2 + \left(3u^{20}\right)^3 + \left(u^{15}\right)^4 = \left(2u^{12}\right)^5. \tag{1.7}$$

A parametrization:

$$a = \left(m(m^{24} + n^{24} + p^{24})\right)^{12} \tag{1.8}$$

$$b = \left(n(m^{24} + n^{24} + p^{24})\right)^{8} \tag{1.9}$$

$$c = \left(p(m^{24} + n^{24} + p^{24})\right)^{6} \tag{1.10}$$

$$d = \left(m^{24} + n^{24} + p^{24}\right)^{5} \tag{1.11}$$

This parametrization gives only a small part of the solutions, but it proves the existence of an infinite number of solutions.

We next consider a similar equation

$$a^2 + b^3 + c^4 + d^5 + e^6 = f^7. \tag{1.12}$$

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Some solutions:

$$2^2 + 3^3 + 1^4 + 2^5 + 2^6 = 2^7 \tag{1.13}$$

$$5^2 + 8^3 + 5^4 + 4^5 + 1^6 = 3^7 \tag{1.14}$$

$$9^2 + 10^3 + 3^4 + 4^5 + 1^6 = 3^7 \tag{1.15}$$

$$5^2 + 23^3 + 8^4 + 2^5 + 2^6 = 4^7 \tag{1.16}$$

$$7^2 + 27^3 + 10^4 + 8^5 + 5^6 = 5^7 \tag{1.17}$$

Parametrization (such exponents make it of little practical use; but it may be useful with modular arithmetic):

$$a = \left(m(m^{350} + n^{350} + p^{350} + q^{350} + r^{350})\right)^{150} \tag{1.18}$$

$$b = \left(n(m^{350} + n^{350} + p^{350} + q^{350} + r^{350})\right)^{100} \tag{1.19}$$

$$c = \left(p(m^{350} + n^{350} + p^{350} + q^{350} + r^{350})\right)^{75} \tag{1.20}$$

$$d = \left(q(m^{350} + n^{350} + p^{350} + q^{350} + r^{350})\right)^{60} \tag{1.21}$$

$$e = \left(r(m^{350} + n^{350} + p^{350} + q^{350} + r^{350})\right)^{50} \tag{1.22}$$

$$f = \left(m^{350} + n^{350} + p^{350} + q^{350} + r^{350}\right)^{43} \tag{1.23}$$

2. Generalization

We can try to generalize our parametrization to an arbitrary equation

$$a_1^{d_1} + a_2^{d_2} + ... + a_n^{d_n} = b^c \tag{2.1}$$

and thus prove that (2.1) has an infinite number of solutions.

To follow the procedure of parametrization of (1.1) and (1.12), we must write a_i and b as

$$a_i = \left(m_i(m_1^k + m_2^k + ... + m_n^k)\right)^{x_i}, \qquad 1 \leq i \leq n \tag{2.2}$$

$$b = \left(m_1^k + m_2^k + ... + m_n^k\right)^w, \tag{2.3}$$

where

$$d_i x_i = k, \qquad 1 \leq i \leq n \tag{2.4}$$

$$cw = k + 1. \tag{2.5}$$

The system of equations (2.4 - 2.5) has solutions for (x_i, k, w), if

$$gcd(lcm(d_1, d_2, ...d_n), c) = 1. \tag{2.6}$$

In this case we have parametrization (2.2 - 2.3), and therefore the equation (2.1) has an infinite number of solutions.

Note, that for $n = 2$, $d_i, c > 2$, the parametrization supports Beal's conjecture.

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